Alex Rosh: Model categories: the sequel

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Def A category $\mathcal{C}$ is <u>homotopical</u> if it has a wide subcat W (containing all objects) with morphisms satisfying the 2 of 6 property

$$\bullet \xrightarrow{f} \bullet \xrightarrow{g} \bullet \xrightarrow{h} \bullet$$

If gf and hg are in W, so are f, g, h and hgf. This implies 2 of 3 property by letting one of them be identity. Every MC is homotopical

Thm (Kan Recognition) Let $\mathcal{M}$ be a bicomplete homotopical category with sets of morphisms I and J that satisfy

(1) I and J both permit the small object argument

(2) $LLP(RLP(J)) \subseteq LLP(RLP(I)) \cap W$

(3) $RLP(I) \subseteq RLP(J) \cap W$

(4) One of (2) or (3) is an equality

Then $\mathcal{M}$ is a CGMC with generating sets I and J.

Thm (Kan Transfer) Let $\mathcal{M}$ be a CGMC with sets I and J and $\mathcal{N}$ a bicomplete category. Assume there is an adjunction $F: \mathcal{M} \rightleftarrows \mathcal{N}: U$

Then if

1) FI and FJ permit the SOA

2) U takes relative FJ-cell complexes to weak equivs

Then $\mathcal{N}$ has a CGMC structure with generating sets FI and FJ.

The weak equivs in $\mathcal{N}$ are maps taken to weak equivs in $\mathcal{M}$.

<u>Def</u> For a set of morphisms I, the <u>subcat of relative I-cell complexes</u> is the subcat of transfinite compositions of pushouts of maps in I.

<u>Bousfield localization</u> ($\mathcal{M}$ is simplicial or topological)

<u>Def</u> Let C be a class of morphisms in a model cat $\mathcal{M}$. A fibrant object W is <u>C-local</u> if any map $f: A \to B$ in C if $\mathcal{M}(B, W) \xrightarrow{f^*} \mathcal{M}(A, W)$ is a weak equiv.

A map $g: X \to Y$ of cofibrant objects is a <u>C-local equiv</u> if

$$\mathcal{M}(Y, W) \xrightarrow{g^*} \mathcal{M}(X, W)$$ is weak equiv for each C-local W.

<u>Def</u> The <u>Bousfield localization</u> of $\mathcal{M}$ at a class of morphisms C is a MC in which (1) weak equivs are C-local equivs, (2) cofibs are as in $\mathcal{M}$ and (3) fibrations are defined by RLP along triv cofib.

Sometimes "Bousfield localization" is the functor

$$L_C (\mathcal{M}, \text{old MC structure}) \to (\mathcal{M}, \text{new MC structure})$$

with nat trans $\eta: 1 \Rightarrow L_C$

Ex $\mathcal{M} = \mathcal{T}op$

$h_* = H(-;\mathbb{Z}_{(p)}) \rightsquigarrow$ p-localization

$h_* = H(-;\mathbb{Z}/p) \rightsquigarrow$ p-adic completion

Def Dn the strict MC structure on simplicial spectra sSp

1. and 2. $X \xrightarrow{f} Y$ is a weak equiv/fibn if $f: X_n \to Y_n$ is $\forall n$

3: $X \to Y$ is a cofib if $f_0 : X_0 \to Y_0$ is one and

pushout corner map

$$X_{n+1} \coprod_{\Sigma X_n} \Sigma Y_n \longrightarrow Y_{n+1}$$ is one for $n \geq 0$

$\left(\pi_n X := \operatorname{colim} \pi_{n+k} X_k\right)$

Def The stable model structure on sSp

1. $f: X \to Y$ is a weak equiv if $\pi_*(f)$ is iso
2. $f: X \to Y$ is a stable cofib if it is a strict cofib
3. $f: X \to Y$ is a stable fib if it is a strict fib and

Assume there is a functor $Q: sSp \to sSp$ and a nat tran $\eta: \mathbb{1} \Rightarrow Q$ s.t. $QX \to QQX$ is weak equiv

e.g. $(QX)_n = \operatorname{colim}_i \operatorname{Sing} \Omega^i |X_{n+i}|$

$f: X \to Y$ is fibration if and

$$\begin{array}{ccc} X_n & \longrightarrow & (QX)_n \\ \downarrow & & \downarrow \\ Y_n & \longrightarrow & (QY)_n \end{array}$$

homotopy pullback

This is what the RLP for fibrations means

Ex Localize $\mathcal{T}$ with respect to the map $S^{n+1} \to *$. Local objects are spaces X with $\Omega^{n+1} X \simeq *$, i.e. $\pi_k X = 0$ for $k \geq n+1$.

The loczn of X is $P^n X$, the nth Postnikov section.

<u>Thm</u> If S is a set and M is either
left proper and cellular or
left proper and combinatorial, then
Bousfield loczn exists

where <u>left proper</u> means

$$\begin{tikzcd} A \arrow[r, "\text{cofib}"] \arrow[d, "\simeq"'] & B \arrow[d, "\simeq"] \\ X \arrow[r] & \text{pushout} \end{tikzcd}$$

We can construct loczn functor by fibrant replacement.